Filomat 29:4 (2015), 687–701
DOI 10.2298/FIL1504687A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some Remarks on Regular Integers Modulo n

Brăduţ Apostol[a], László Tóth[b]

[a]Pedagogic High School "Spiru Haret", Str. Timotei Cipariu, RO-620004 Focşani, Romania
[b]Institute of Mathematics, Universität für Bodenkultur, Gregor Mendel-Straße 33, A-1180 Vienna, Austria
Department of Mathematics, University of Pécs, Ifjúság u. 6, H-7624 Pécs, Hungary

Abstract. An integer k is called regular (mod n) if there exists an integer x such that $k^2x \equiv k \pmod{n}$. This holds true if and only if k possesses a weak order (mod n), i.e., there is an integer $m \geq 1$ such that $k^{m+1} \equiv k \pmod{n}$. Let $\varrho(n)$ denote the number of regular integers (mod n) in the set $\{1, 2, \ldots, n\}$. This is an analogue of Euler's ϕ function. We introduce the multidimensional generalization of ϱ, which is the analogue of Jordan's function. We establish identities for the power sums of regular integers (mod n) and for some other finite sums and products over regular integers (mod n), involving the Bernoulli polynomials, the Gamma function and the cyclotomic polynomials, among others. We also deduce an analogue of Menon's identity and investigate the maximal orders of certain related functions.

1. Introduction

Throughout the paper we use the notations: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{Z}$ is the set of integers, $\lfloor x \rfloor$ is the integer part of x, $\mathbf{1}$ is the function given by $\mathbf{1}(n) = 1$ $(n \in \mathbb{N})$, id is the function given by $\text{id}(n) = n$ $(n \in \mathbb{N})$, ϕ is Euler's totient function, $\tau(n)$ is the number of divisors of n, μ is the Möbius function, $\omega(n)$ stands for the number of prime factors of n, Λ is the von Mangoldt function, $\kappa(n) := \prod_{p \mid n} p$ is the largest squarefree divisor of n, $c_n(t)$ are the Ramanujan sums defined by $c_n(t) := \sum_{1 \leq k \leq n, \gcd(k,n)=1} \exp(2\pi i k t/n)$ $(n \in \mathbb{N}, t \in \mathbb{Z})$, ζ is the Riemann zeta function. Other notations will be fixed inside the paper.

Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then k is called regular (mod n) if there exists $x \in \mathbb{Z}$ such that $k^2x \equiv k$ (mod n). This holds true if and only if k possesses a weak order (mod n), i.e., there is $m \in \mathbb{N}$ such that $k^{m+1} \equiv k$ (mod n). Every $k \in \mathbb{Z}$ is regular (mod 1). If $n > 1$ and its prime power factorization is $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$, then k is regular (mod n) if and only if for every $i \in \{1, \ldots, r\}$ either $p_i \nmid k$ or $p_i^{\nu_i} \mid k$. Also, k is regular (mod n) if and only if $\gcd(k, n)$ is a unitary divisor of n. We recall that d is said to be a unitary divisor of n if $d \mid n$ and $\gcd(d, n/d) = 1$, notation $d \parallel n$. Note that if n is squarefree, then every $k \in \mathbb{Z}$ is regular (mod n). See the papers [1, 14, 15, 20] for further discussion and properties of regular integers (mod n), and their connection with the notion of regular elements of a ring in the sense of J. von Neumann.

An integer k is regular (mod n) if and only if $k + n$ is regular (mod n). Therefore, it is justified to consider the set

$$\text{Reg}_n := \{k \in \mathbb{N} : 1 \leq k \leq n, k \text{ is regular (mod } n)\}$$

2010 *Mathematics Subject Classification*. Primary 11A25; Secondary 11B68, 11N37, 33B10, 33B15
Keywords. regular integer (mod n), Euler's totient function, Jordan's function, Ramanujan's sum, unitary divisor, Bernoulli numbers and polynomials, Gamma function, finite trigonometric sums and products, cyclotomic polynomial
Received: 07 October 2013; Accepted: 24 February 2014
Communicated by Hari M. Srivastava
L. Tóth gratefully acknowledges support from the Austrian Science Fund (FWF) under the project Nr. M1376-N18.
Email addresses: apo_brad@yahoo.com (Brăduţ Apostol), ltoth@gamma.ttk.pte.hu (László Tóth)

and the quantity $\varrho(n) := \# \mathrm{Reg}_n$. For example, $\mathrm{Reg}_{12} = \{1,3,4,5,7,8,9,11,12\}$ and $\varrho(12) = 9$. If n is squarefree, then $\mathrm{Reg}_n = \{1,2,\ldots,n\}$ and $\varrho(n) = n$. Note that $1, n \in \mathrm{Reg}_n$ for every $n \in \mathbb{N}$. The arithmetic function ϱ is an analogue of Euler's ϕ function, it is multiplicative and $\varrho(p^\nu) = \phi(p^\nu) + 1 = p^\nu - p^{\nu-1} + 1$ for every prime power p^ν ($\nu \in \mathbb{N}$). Consequently,

$$\varrho(n) = \sum_{d \,\|\, n} \phi(d) \quad (n \in \mathbb{N}). \tag{1}$$

See, e.g., [13] for general properties of unitary divisors, in particular the unitary convolution of the arithmetic functions f and g defined by $(f \times g)(n) = \sum_{d \,\|\, n} f(d) g(n/d)$. Here $f \times g$ preserves the multiplicativity of the functions f and g. We refer to [20] for asymptotic properties of the function ϱ.

The function

$$\overline{c}_n(t) := \sum_{k \in \mathrm{Reg}_n} \exp(2\pi i k t/n) \quad (n \in \mathbb{N}, t \in \mathbb{Z}),$$

representing an analogue of the Ramanujan sum $c_n(t)$ was investigated in the paper [8]. We have

$$\overline{c}_n(t) = \sum_{d \,\|\, n} c_d(t) \quad (n \in \mathbb{N}, t \in \mathbb{Z}).$$

It turns out that for every fixed t the function $n \mapsto \overline{c}_n(t)$ is multiplicative, $\overline{c}_n(0) = \varrho(n)$ and $\overline{c}_n(1) = \overline{\mu}(n)$ is the characteristic function of the squarefull integers n.

The gcd-sum function is defined by $P(n) := \sum_{k=1}^{n} \gcd(k,n) = \sum_{d \mid n} d\, \phi(n/d)$, see [22]. The following analogue of the gcd-sum function was introduced in the paper [21]:

$$\widetilde{P}(n) := \sum_{k \in \mathrm{Reg}_n} \gcd(k,n).$$

One has

$$\widetilde{P}(n) = \sum_{d \,\|\, n} d\, \phi(n/d) = n \prod_{p \mid n} \left(2 - \frac{1}{p}\right) \quad (n \in \mathbb{N}),$$

the asymptotic properties of $\widetilde{P}(n)$ being investigated in [10, 22, 26, 27].

In the present paper we discuss some further properties of the regular integers (mod n). We first introduce the multidimensional generalization ϱ_r ($r \in \mathbb{N}$) of the function ϱ, which is the analogue of the Jordan function ϕ_r, where $\phi_r(n)$ is defined as the number of ordered r-tuples $(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r$ such that $\gcd(k_1, \ldots, k_r)$ is prime to n (see, e.g., [13, 18]). Then we consider the sum $\mathrm{S[reg]}_r(n)$ of r-th powers of the regular integers (mod n) belonging to Reg_n. In the case $r \in \mathbb{N}$ we deduce an exact formula for $\mathrm{S[reg]}_r(n)$ involving the Bernoulli numbers B_m. For a positive real number r we derive an asymptotic formula for $\mathrm{S[reg]}_r(n)$. We combine the functions $\overline{c}_n(t)$ and $\widetilde{P}(n)$ defined above and establish identities for sums, respectively products over the integers in Reg_n concerning the Bernoulli polynomials $B_m(x)$, the Gamma function Γ, the cyclotomic polynomials $\Phi_m(x)$ and certain trigonometric functions. We point out that for n squarefree these identities reduce to the corresponding ones over $\{1,2,\ldots,n\}$. We also deduce an analogue of Menon's identity and investigate the maximal orders of some related functions.

2. A Generalization of the Function ϱ

For $r \in \mathbb{N}$ let $\varrho_r(n)$ be the number of ordered r-tuples $(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r$ such that $\gcd(k_1, \ldots, k_r)$ is regular (mod n). If $r = 1$, then $\varrho_1 = \varrho$. The arithmetic function ϱ_r is the analogue of the Jordan function ϕ_r, defined in the Introduction and verifying $\phi_r(n) = n^r \prod_{p \mid n}(1 - 1/p^r)$ ($n \in \mathbb{N}$).

Proposition 2.1. *i) For every $r, n \in \mathbb{N}$,*

$$\varrho_r(n) = \sum_{d \,\|\, n} \phi_r(d).$$

ii) The function ϱ_r is multiplicative and for every prime power p^ν ($\nu \in \mathbb{N}$),

$$\varrho_r(p^\nu) = p^{r\nu} - p^{r(\nu-1)} + 1.$$

Proof. i) The integer $\gcd(k_1, \ldots, k_r)$ is regular (mod n) if and only if $\gcd(\gcd(k_1, \ldots, k_r), n) \,\|\, n$, that is $\gcd(k_1, \ldots, k_r, n) \,\|\, n$ and grouping the r-tuples $(k_1, \ldots, k_r)$ according to the values $\gcd(k_1, \ldots, k_r, n) = d$ we deduce that

$$\varrho_r(n) = \sum_{\substack{(k_1,\ldots,k_r)\in\{1,\ldots,n\}^r \\ \gcd(k_1,\ldots,k_r) \text{ regular (mod } n)}} 1 = \sum_{d \,\|\, n} \sum_{\substack{(k_1,\ldots,k_r)\in\{1,\ldots,n\}^r \\ \gcd(k_1,\ldots,k_r,n)=d}} 1$$

$$= \sum_{d \,\|\, n} \sum_{\substack{(\ell_1,\ldots,\ell_r)\in\{1,\ldots,n/d\}^r \\ \gcd(\ell_1,\ldots,\ell_r,n/d)=1}} 1,$$

where the inner sum is $\phi_r(n/d)$, according to its definition.

ii) Follows at once by i). □

More generally, for a fixed real number s let $\phi_s(n) = \sum_{d \mid n} d^s \mu(n/d)$ be the generalized Jordan function and define ϱ_s by

$$\varrho_s(n) = \sum_{d \,\|\, n} \phi_s(d) \quad (n \in \mathbb{N}). \tag{2}$$

The functions ϕ_s and ϱ_s (which will be used in the next results of the paper) are multiplicative and for every prime power p^ν ($\nu \in \mathbb{N}$) one has $\phi_s(p^\nu) = p^{s\nu} - p^{s(\nu-1)}$ and $\varrho_s(p^\nu) = p^{s\nu} - p^{s(\nu-1)} + 1$. Note that $\phi_{-s}(n) = n^{-s} \prod_{p^\nu \,\|\, n} (1 - p^s)$ and $\varrho_{-s}(n) = n^{-s} \prod_{p^\nu \,\|\, n} (p^{s\nu} - p^s + 1)$.

Proposition 2.2. *If $s > 1$ is a real number, then*

$$\sum_{n \le x} \varrho_s(n) = \frac{x^{s+1}}{s+1} \prod_p \left(1 - \frac{1}{p^{s+1}} + \frac{p-1}{p(p^{s+1}-1)}\right) + O(x^s). \tag{3}$$

Proof. We need the following asymptotics. Let $s > 0$ be fixed real number. Then uniformly for real $x > 1$ and $t \in \mathbb{N}$,

$$\phi_s(x,t) := \sum_{\substack{n \le x \\ \gcd(n,t)=1}} \phi_s(n) = \frac{x^{s+1}}{(s+1)\zeta(s+1)} \cdot \frac{t^s \phi(t)}{\phi_{s+1}(t)} + O(x^s 2^{\omega(t)}). \tag{4}$$

To show (4) use the known estimate, valid for every fixed $s > 0$ and $t \in \mathbb{N}$,

$$\sum_{\substack{n \le x \\ \gcd(n,t)=1}} n^s = \frac{x^{s+1}}{s+1} \cdot \frac{\phi(t)}{t} + O\left(x^s 2^{\omega(t)}\right). \tag{5}$$

We obtain

$$\phi_s(x,t) = \sum_{\substack{de=n \le x \\ \gcd(n,t)=1}} \mu(d) e^s = \sum_{\substack{d \le x \\ \gcd(d,t)=1}} \mu(d) \sum_{\substack{e \le x/d \\ \gcd(e,t)=1}} e^s$$

$$= \sum_{\substack{d\le x\\ \gcd(d,t)=1}} \mu(d)\left(\frac{(x/d)^{s+1}}{s+1}\cdot\frac{\phi(t)}{t} + O\left((x/d)^s 2^{\omega(t)}\right)\right)$$

$$= \frac{x^{s+1}}{s+1}\cdot\frac{\phi(t)}{t}\sum_{\substack{d=1\\ \gcd(d,t)=1}}^{\infty}\frac{\mu(d)}{d^{s+1}} + O\left(x^{s+1}\sum_{d>x}\frac{1}{d^{s+1}}\right) + O\left(x^s 2^{\omega(t)}\right),$$

giving (4). Now from (2) and (4),

$$\sum_{n\le x}\varrho_s(n) = \sum_{\substack{de=n\le x\\ \gcd(d,e)=1}}\phi_s(e) = \sum_{d\le x}\sum_{\substack{e\le x/d\\ \gcd(e,d)=1}}\phi_s(e) = \sum_{d\le x}\phi_s(x/d,d)$$

$$= \frac{x^{s+1}}{(s+1)\zeta(s+1)}\sum_{d=1}^{\infty}\frac{\phi(d)}{d\phi_{s+1}(d)} + O\left(x^{s+1}\sum_{d>x}\frac{\phi(d)}{d\phi_{s+1}(d)}\right) + O\left(x^s\sum_{d\le x}\frac{2^{\omega(d)}}{d^s}\right),$$

and for $s>1$ this leads to (3). □

Compare (3) to the corresponding formula for the Jordan function ϕ_s, i.e., to (4) with $t=1$.

Remark 2.3. For the function ϱ one has

$$\sum_{n\le x}\varrho(n) = \frac{1}{2}\prod_p\left(1-\frac{1}{p^2(p+1)}\right)x^2 + R(x),$$

where $R(x)=O(x\log^3 x)$ can be obtained by the elementary arguments given above. This can be improved into $R(x)=O(x\log x)$ using analytic methods. See [20] for references.

3. A General Scheme

In order to give exact formulas for certain sums and products over the regular integers (mod n) we first present a simple result for a general sum over Reg_n, involving a weight function w and an arithmetic function f. It would be possible to consider a more general sum, namely over the ordered r-tuples $(k_1,\ldots,k_r)\in\{1,\ldots,n\}^r$ such that $\gcd(k_1,\ldots,k_r)$ is regular (mod n), but we confine ourselves to the following result. See [24] for another similar scheme concerning weighted gcd-sum functions.

Proposition 3.1. *i) Let $w:\mathbb{N}^2\to\mathbb{C}$ and $f:\mathbb{N}\to\mathbb{C}$ be arbitrary functions and consider the sum*

$$R_{w,f}(n) := \sum_{k\in\mathrm{Reg}_n} w(k,n)f(\gcd(k,n)).$$

Then

$$R_{w,f}(n) = \sum_{d\,\|\,n} f(d)\sum_{\substack{j=1\\ \gcd(j,n/d)=1}}^{n/d} w(dj,n) \quad (n\in\mathbb{N}). \tag{6}$$

ii) Assume that there is a function $g:(0,1]\to\mathbb{C}$ such that $w(k,n)=g(k/n)$ $(1\le k\le n)$ and let

$$\overline{G}(n) = \sum_{\substack{k=1\\ \gcd(k,n)=1}}^{n} g(k/n) \quad (n\in\mathbb{N}).$$

Then

$$R_{w,f}(n) = \sum_{d\,\|\,n} f(d)\overline{G}(n/d) \quad (n\in\mathbb{N}). \tag{7}$$

Proof. i) Using that k is regular (mod n) if and only if $\gcd(k,n) \parallel n$ and grouping the terms according to the values of $\gcd(k,n) = d$ and denoting $k = dj$ we have at once

$$R_{w,f}(n) = \sum_{d \parallel n} f(d) \sum_{\substack{k=1 \\ \gcd(k,n)=d}}^{n} w(k,n) = \sum_{d \parallel n} f(d) \sum_{\substack{j=1 \\ \gcd(j,n/d)=1}}^{n/d} w(dj,n).$$

ii) Now

$$\sum_{j=1}^{n/d} w(dj,n) = \sum_{j=1}^{n/d} g(j/(n/d)) = \overline{G}(n/d).$$

□

Remark 3.2. For the function g given above let

$$G(n) := \sum_{k=1}^{n} g(k/n).$$

Then we have

$$\overline{G}(n) = \sum_{d \mid n} \mu(d) G(n/d) \quad (n \in \mathbb{N}). \tag{8}$$

Indeed, as it is well known, $\overline{G}(n) = \sum_{k=1}^{n} g(k/n) \sum_{d \mid \gcd(k,n)} \mu(d)$, giving (8).

4. Power Sums of Regular Integers (mod *n*)

In this section we investigate the sum of r-th powers ($r \in \mathbb{N}$) of the regular integers (mod n). Let B_m ($m \in \mathbb{N}_0$) be the Bernoulli numbers defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

Here $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_m = 0$ for every $m \geq 3$, m odd and one has the recurrence relation

$$B_m = \sum_{j=0}^{m} \binom{m}{j} B_j \quad (m \geq 2). \tag{9}$$

It is well known that for every $n, r \in \mathbb{N}$,

$$S_r(n) := \sum_{k=1}^{n} k^r = \frac{1}{r+1} \sum_{m=0}^{r} (-1)^m \binom{r+1}{m} B_m n^{r+1-m}$$

$$= \frac{n^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} n^{r+1-2m}. \tag{10}$$

From here one obtains, using the same device as that given in Remark 3.2 that for every $n, r \in \mathbb{N}$ with $n \geq 2$,

$$\text{S[relpr]}_r(n) := \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k^r = \frac{n^r}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \phi_{1-2m}(n), \tag{11}$$

where $\phi_{1-2m}(n) = n^{1-2m} \prod_{p\mid n}\left(1-p^{2m-1}\right)$. Formula (11) was given in [19]. Here we prove the following result.

Proposition 4.1. *For every $n, r \in \mathbb{N}$,*

$$\mathrm{S[reg]}_r(n) := \sum_{k\in \mathrm{Reg}_n} k^r = \frac{n^r}{2} + \frac{n^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m}\, \varrho_{1-2m}(n), \tag{12}$$

where

$$\varrho_{1-2m}(n) = n^{1-2m}\prod_{p^\nu \,\|\, n}\left(p^{(2m-1)\nu} - p^{2m-1} + 1\right)$$

is the generalized ϱ function, discussed in Section 2.

Proof. Applying (6) for $w(k,n) = k^r$ and $f = \mathbf{1}$ we have

$$\mathrm{S[reg]}_r(n) = \sum_{d\,\|\,n}\ \sum_{\substack{j=1\\ \gcd(j,n/d)=1}}^{n/d} (dj)^r = \sum_{d\,\|\,n} d^r\, \mathrm{S[relpr]}_r(n/d).$$

Now by (11) we deduce

$$\mathrm{S[reg]}_r(n) = n^r + \sum_{\substack{d\,\|\,n\\ d<n}} d^r \left(\frac{(n/d)^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m}\phi_{1-2m}(n/d)\right)$$

$$= n^r + \frac{n^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m}\sum_{\substack{d\,\|\,n\\ d<n}}\phi_{1-2m}(n/d)$$

$$= n^r + \frac{n^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m}\sum_{\substack{d\,\|\,n\\ d>1}}\phi_{1-2m}(d)$$

$$= n^r - \frac{n^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m} + \frac{n^r}{r+1}\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m}\sum_{d\,\|\,n}\phi_{1-2m}(d).$$

Here $\sum_{d\,\|\,n}\phi_{1-2m}(n) = \varrho_{1-2m}(d)$ by (2). Also, by (9),

$$\sum_{m=0}^{\lfloor r/2\rfloor}\binom{r+1}{2m} B_{2m} = \frac{r+1}{2}$$

and this completes the proof. □

For example, in the cases $r = 1, 2, 3, 4$ we deduce that for every $n \in \mathbb{N}$,

$$\mathrm{S[reg]}_1(n) = \frac{n(\varrho(n)+1)}{2}, \tag{13}$$

$$\mathrm{S[reg]}_2(n) = \frac{n^2}{2} + \frac{n^2\varrho(n)}{3} + \frac{n}{6}\prod_{p^\nu\,\|\,n}\left(p^\nu - p + 1\right), \tag{14}$$

$$\mathrm{S[reg]}_3(n) = \frac{n^3}{2} + \frac{n^3 \varrho(n)}{4} + \frac{n^2}{4} \prod_{p^\nu \,\|\, n} (p^\nu - p + 1),$$

$$\mathrm{S[reg]}_4(n) = \frac{n^4}{2} + \frac{n^4 \varrho(n)}{5} + \frac{n^3}{3} \prod_{p^\nu \,\|\, n} (p^\nu - p + 1) - \frac{n}{30} \prod_{p^\nu \,\|\, n} \left(p^{3\nu} - p^3 + 1\right).$$

The formula (13) was obtained in [20, Th. 3] and [3, Sec. 2], while (14) was given in a different form in [3, Prop. 1]. Note that if n is squarefree, then (12) reduces to (10).

For a real number s consider now the slightly more general sum

$$\mathrm{S[reg]}_s(n, x) := \sum_{\substack{k \le x \\ k \text{ regular (mod } n)}} k^s.$$

Proposition 4.2. *Let $s \ge 0$ be a fixed real number. Then uniformly for real $x > 1$ and $n \in \mathbb{N}$,*

$$\mathrm{S[reg]}_s(n, x) = \frac{x^{s+1}}{s+1} \cdot \frac{\varrho(n)}{n} + O\left(x^s 3^{\omega(n)}\right).$$

Proof. Similar to the proof of Proposition 3.1,

$$\mathrm{S[reg]}_s(n, x) = \sum_{\substack{k \le x \\ \gcd(k,n) \,\|\, n}} k^s = \sum_{d \,\|\, n} d^s \sum_{\substack{j \le x/d \\ \gcd(j, n/d) = 1}} j^s.$$

Now using the estimate (5) we deduce

$$\mathrm{S[reg]}_s(n, x) = \sum_{d \,\|\, n} d^s \left(\frac{(x/d)^{s+1} \phi(n/d)}{(s+1)(n/d)} + O\left((x/d)^s 2^{\omega(n/d)}\right) \right)$$

$$= \frac{x^{s+1}}{(s+1)n} \sum_{d \,\|\, n} \phi(n/d) + O\left(x^s \sum_{d \,\|\, n} 2^{\omega(n/d)} \right),$$

and using (1) the proof is complete. □

5. Identities for other Sums and Products Over Regular Integers (mod *n*)

5.1. Sums Involving Bernoulli Polynomials

Let $B_m(x)$ ($m \in \mathbb{N}_0$) be the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

Here $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $B_3(x) = x^3 - 3x^2/2 + x/2$, $B_m(0) = B_m$ ($m \in \mathbb{N}_0$) are the Bernoulli numbers already defined in Section 4 and one has the recurrence relation

$$B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_j x^{m-j} \quad (m \in \mathbb{N}_0).$$

It is well known (see, e.g., [5, Sect. 9.1]) that for every $n, m \in \mathbb{N}$, $m \ge 2$,

$$T_m(n) := \sum_{k=1}^{n} B_m(k/n) = \frac{B_m}{n^{m-1}}. \tag{15}$$

Furthermore, applying (8) one obtains from (15) that for every $n, m \in \mathbb{N}$, $m \geq 2$,

$$\mathrm{T[relpr]}_m(n) := \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} B_m(k/n) = B_m \phi_{1-m}(n), \tag{16}$$

where $\phi_{1-m}(n) = n^{1-m} \prod_{p \mid n} \left(1 - p^{m-1}\right)$. See [5, Sect. 9.9, Ex. 7]. We now show the validity of the next formula:

Proposition 5.1.1 *For every $n, m \in \mathbb{N}$, $m \geq 2$,*

$$\mathrm{T[reg]}_m(n) := \sum_{k \in \mathrm{Reg}_n} B_m(k/n) = B_m \varrho_{1-m}(n), \tag{17}$$

where $\varrho_{1-m}(n) = n^{1-m} \prod_{p^\nu \parallel n} \left(p^{(m-1)\nu} - p^{m-1} + 1\right)$.

Proof. Choosing $g(x) = B_m(x)$ and $f = \mathbf{1}$ we deduce from (7) by using (16) that

$$\mathrm{T[reg]}_m(n) = \sum_{d \parallel n} \mathrm{T[relpr]}_m(d)$$

$$= B_m \sum_{d \parallel n} \phi_{1-m}(d) = B_m \varrho_{1-m}(n),$$

according to (2). □

Remark 5.1.2 In the case $m = 1$ a direct computation and (13) show that $\mathrm{T[reg]}_1(n) = 1/2$. Also, (17) can be put in the form

$$\sum_{\substack{k=0 \\ k \text{ regular} \pmod n}}^{n-1} B_m(k/n) = B_m \varrho_{1-m}(n),$$

which holds true for every $n, m \in \mathbb{N}$, also for $m = 1$.

5.2. Sums Involving gcd's and the exp Function

Consider in what follows the function

$$\mathrm{P[reg]}_{f,t}(n) := \sum_{k \in \mathrm{Reg}_n} f(\gcd(k,n)) \exp(2\pi i k t/n) \quad (n \in \mathbb{N}, t \in \mathbb{Z}),$$

where f is an arbitrary arithmetic function. For $t = 0$ and $f(n) = n$ ($n \in \mathbb{N}$) we reobtain the function $\widetilde{P}(n)$ and for $f = \mathbf{1}$ we have $\overline{c}_n(t)$, the analogue of the Ramanujan sums, both given in the Introduction. We have

Proposition 5.2.1 *For every f and every $n \in \mathbb{N}$ and $t \in \mathbb{Z}$,*

$$\mathrm{P[reg]}_{f,t}(n) = \sum_{d \parallel n} f(d) c_{n/d}(t).$$

If f is integer valued and multiplicative (in particular, if $f = \mathrm{id}$), then $n \mapsto \mathrm{P[reg]}_{f,t}(n)$ also has these properties.

Proof. Choosing $g(x) = \exp(2\pi i t x)$ from (7) we deduce at once that

$$\mathrm{P[reg]}_{f,t}(n) = \sum_{d \parallel n} f(d) \sum_{\substack{j=1 \\ \gcd(j,n/d)=1}}^{n/d} \exp(2\pi i j t/(n/d)) = \sum_{d \parallel n} f(d) c_{n/d}(t).$$

□

For $t = 1$ and $f = \mathrm{id}$ this gives the multiplicative function

$$\mathrm{P[reg]}_{\mathrm{id},1}(n) = \sum_{d \,\|\, n} d\mu(n/d),$$

not investigated in the literature, as far as we know. Here $\mathrm{P[reg]}_{\mathrm{id},1}(p^\nu) = p - 1$ for every prime p and $\mathrm{P[reg]}_{\mathrm{id},1}(p^\nu) = p^\nu$ for every prime power p^ν with $\nu \geq 2$.

Proposition 5.2.2 *We have*

$$\sum_{n \leq x} \mathrm{P[reg]}_{\mathrm{id},1}(n) = \frac{x^2}{2} \prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^3}\right) + O(x \log^2 x).$$

Proof. Using (5) for $s = 1$ we deduce

$$\sum_{n \leq x} \mathrm{P[reg]}_{\mathrm{id},1}(n) = \sum_{d \leq x} \mu(d) \sum_{\substack{\delta \leq x/d \\ \gcd(\delta,d)=1}} \delta$$

$$= \sum_{d \leq x} \mu(d) \left(\frac{\phi(d)(x/d)^2}{2d} + O((x/d) 2^{\omega(d)}) \right)$$

$$= \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{\mu(d)\phi(d)}{d^3} + O\left(x^2 \sum_{d > x} \frac{1}{d^2} \right) + O\left(x \sum_{d \leq x} \frac{2^{\omega(d)}}{d} \right),$$

giving the result. □

5.3. An Analogue of Menon's Identity

Our next result is the analogue of Menon's identity ([12], see also [23])

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(k-1, n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}). \tag{18}$$

Proposition 5.3.1 *For every* $n \in \mathbb{N}$*,*

$$\sum_{k \in \mathrm{Reg}_n} \gcd(k-1, n) = \sum_{d \,\|\, n} \phi(d)\tau(d) = \prod_{p^\nu \,\|\, n} \left(p^{\nu-1}(p-1)(\nu+1) + 1 \right).$$

Proof. Applying (6) for $w(k, n) = \gcd(k-1, n)$ and $f = \mathbf{1}$ we deduce

$$S_n := \sum_{k \in \mathrm{Reg}_n} \gcd(k-1, n) = \sum_{d \,\|\, n} \sum_{\substack{j=1 \\ \gcd(j,n/d)=1}}^{n/d} \gcd(dj-1, n)$$

$$= \sum_{d \,\|\, n} \sum_{\substack{j=1 \\ \gcd(j,n/d)=1}}^{n/d} \gcd(dj-1, n/d),$$

since $\gcd(dj-1, d) = 1$ for every d and j. Now we use the identity

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \gcd(ak-1, n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}),$$

valid for every fixed $a \in \mathbb{N}$ with $\gcd(a, n) = 1$, see [23, Cor. 14] (for $a = 1$ this reduces to (18)). Choose $a = d$. Since $d \parallel n$ we have $\gcd(d, n/d) = 1$ and obtain

$$S_n = \sum_{d \parallel n} \phi(n/d)\tau(n/d) = \sum_{d \parallel n} \phi(d)\tau(d).$$

□

5.4. **Trigonometric Sums**

Further identities for sums over Reg_n can be derived. As examples, consider the following known trigonometric identities. For every $n \in \mathbb{N}$, $n \geq 2$,

$$\sum_{k=1}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{n}{2};$$

furthermore, for every $n \in \mathbb{N}$ odd number,

$$\sum_{k=1}^{n} \tan^2\left(\frac{k\pi}{n}\right) = n^2 - n;$$

and also for every $n \in \mathbb{N}$ odd,

$$\sum_{k=1}^{n} \tan^4\left(\frac{k\pi}{n}\right) = \frac{1}{3}(n^4 - 4n^2 + 3n).$$

See, for example, [4] for a discussion and proofs of these identities. See [16, Appendix 3] for other similar identities. By the approach given in Remark 3.2 we deduce that for every $n \in \mathbb{N}$,

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{\phi(n) + \mu(n)}{2};$$

for every $n \in \mathbb{N}$ odd number,

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \tan^2\left(\frac{k\pi}{n}\right) = \phi_2(n) - \phi(n);$$

and for every $n \in \mathbb{N}$ odd,

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \tan^4\left(\frac{k\pi}{n}\right) = \frac{1}{3}(\phi_4(n) - 4\phi_2(n) + 3\phi(n)).$$

This gives the next results. The proof is similar to the proofs given above.

Proposition 5.4.1 *For every $n \in \mathbb{N}$,*

$$\sum_{k \in \mathrm{Reg}_n} \cos^2\left(\frac{k\pi}{n}\right) = \frac{\varrho(n) + \overline{\mu}(n)}{2},$$

where $\overline{\mu}(n) = \sum_{d \parallel n} \mu(d)$ is the characteristic function of the squarefull integers n, given in the Introduction.

Proposition 5.4.2 *For every $n \in \mathbb{N}$ odd number,*

$$\sum_{k\in \mathrm{Reg}_n} \tan^2\left(\frac{k\pi}{n}\right) = \varrho_2(n) - \varrho(n),$$

$$\sum_{k\in \mathrm{Reg}_n} \tan^4\left(\frac{k\pi}{n}\right) = \frac{1}{3}(\varrho_4(n) - 4\varrho_2(n) + 3\varrho(n)).$$

5.5. **The product of numbers in** Reg_n

It is known (see, e.g., [16, p. 197, Ex. 24]) that for every $n \in \mathbb{N}$,

$$\mathrm{Q[relpr]}(n) := \prod_{\substack{k=1\\ \gcd(k,n)=1}}^{n} k = n^{\phi(n)}A(n), \tag{19}$$

where

$$A(n) = \prod_{d\,|\,n} (d!/d^d)^{\mu(n/d)}.$$

We show that

Proposition 5.5.1 *For every $n \in \mathbb{N}$,*

$$\mathrm{Q[reg]}(n) := \prod_{k\in \mathrm{Reg}_n} k = n^{\varrho(n)} \prod_{d\,\|\,n} A(d).$$

Proof. Choosing $w(k,n) = \log k$ and $f = \mathbf{1}$ in Proposition 3.1 we have

$$\log \mathrm{Q[reg]}(n) = \sum_{k\in \mathrm{Reg}_n} \log k = \sum_{d\,\|\,n} \sum_{\substack{j=1\\ \gcd(j,n/d)=1}}^{n/d} \log(dj)$$

$$= \sum_{d\,\|\,n} \Big(\phi(n/d)\log d + \log \mathrm{Q[relpr]}(n/d)\Big)$$

$$= \sum_{d\,\|\,n} \Big(\phi(d)\log(n/d) + \log \mathrm{Q[relpr]}(d)\Big)$$

$$= (\log n)\sum_{d\,\|\,n}\phi(d) - \sum_{d\,\|\,n}\phi(d)\log d + \sum_{d\,\|\,n}\log \mathrm{Q[relpr]}(d).$$

Hence,

$$\mathrm{Q[reg]}(n) = n^{\varrho(n)} \prod_{d\,\|\,n} \frac{\mathrm{Q[relpr]}(d)}{d^{\phi(d)}}.$$

Now the result follows from the identity (19). □

5.6. **Products Involving the Gamma Function**

Let Γ be the Gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

It is well known that for every $n \in \mathbb{N}$,

$$R(n) := \prod_{k=1}^{n} \Gamma(k/n) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}, \tag{20}$$

which is a consequence of Gauss' multiplication formula. For the q-analogs of the Gamma and Beta functions and the multiplication formula see the recent papers [6, 7] published in this journal. Furthermore, for every $n \in \mathbb{N}$, $n \geq 2$,

$$\mathrm{R[relpr]}(n) := \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \Gamma(k/n) = \frac{(2\pi)^{\phi(n)/2}}{\exp(\Lambda(n)/2)}, \tag{21}$$

see [11, 17].

Proposition 5.6.1 *For every $n \in \mathbb{N}$,*

$$\mathrm{R[reg]}(n) := \prod_{k \in \mathrm{Reg}_n} \Gamma(k/n) = \frac{(2\pi)^{(\varrho(n)-1)/2}}{\sqrt{\kappa(n)}}. \tag{22}$$

Proof. Choosing $g = \log \Gamma$ and $f = \mathbf{1}$ in (7) and using (21) we deduce

$$\log \mathrm{R[reg]}(n) = \sum_{k \in \mathrm{Reg}_n} \log \Gamma(k/n) = \sum_{d \,\|\, n} \log \mathrm{R[relpr]}(d)$$

$$= \sum_{\substack{d \,\|\, n \\ d>1}} \left(\frac{\log 2\pi}{2} \phi(d) - \frac{1}{2}\Lambda(d) \right)$$

$$= \sum_{d \,\|\, n} \left(\frac{\log 2\pi}{2} \phi(d) - \frac{1}{2}\Lambda(d) \right) - \frac{\log 2\pi}{2} = \frac{\log 2\pi}{2}(\varrho(n) - 1) - \frac{1}{2} \sum_{d \,\|\, n} \Lambda(d),$$

where the last sum is $\log \kappa(n)$. □

For squarefree n (22) reduces to (20).

5.7. **Identities Involving Cyclotomic Polynomials**

Let $\Phi_n(x)$ ($n \in \mathbb{N}$) stand for the cyclotomic polynomials (see, e.g., [9, Ch. 13]) defined by

$$\Phi_n(x) = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} (x - \exp(2\pi i k/n)).$$

Consider now the following analogue of the cyclotomic polynomials $\Phi_n(x)$:

$$\Phi[\mathrm{reg}]_n(x) = \prod_{k \in \mathrm{Reg}_n} (x - \exp(2\pi i k/n)).$$

The application of Proposition 3.1 gives the following result.

Proposition 5.7.1 *For every $n \in \mathbb{N}$,*

$$\Phi[\text{reg}]_n(x) = \prod_{d \parallel n} \Phi_d(x).$$

Here the degree of $\Phi[\text{reg}]_n(x)$ is $\varrho(n)$. If n is squarefree, then $\Phi[\text{reg}]_n(x) = x^n - 1$ and for example, $\Phi[\text{reg}]_{12}(x) = \Phi_1(x)\Phi_3(x)\Phi_4(x)\Phi_{12}(x) = x^9 - x^6 + x^3 - 1$.

It is well known that for every $n \in \mathbb{N}$, $n \geq 2$,

$$U(n) := \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \sin\left(\frac{k\pi}{n}\right) = \frac{\Phi_n(1)}{2^{\phi(n)}}, \tag{23}$$

where

$$\Phi_n(1) = \begin{cases} p, & n = p^\nu, \nu \geq 1, \\ 1, & \text{otherwise, i.e., if } \omega(n) \geq 2, \end{cases}$$

and for $n \geq 3$,

$$V(n) := \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \cos\left(\frac{k\pi}{n}\right) = \frac{\Phi_n(-1)}{(-4)^{\phi(n)/2}}, \tag{24}$$

where

$$\Phi_n(-1) = \begin{cases} 2, & n = 2^\nu, \\ p, & n = 2p^\nu, p > 2 \text{ prime}, \nu \geq 1, \\ 1, & \text{otherwise.} \end{cases}$$

For every $n \in \mathbb{N}$, $\prod_{k \in \text{Reg}_n} \sin(k\pi/n) = 0$, since $n \in \text{Reg}_n$. This suggests to consider also the modified products

$$\text{U[regmod]}(n) := \prod_{\substack{k=1 \\ k \text{ regular } (\text{mod } n)}}^{n-1} \sin\left(\frac{k\pi}{n}\right),$$

$$\text{V[regmod]}(n) := \prod_{\substack{k=1 \\ k \text{ regular } (\text{mod } n)}}^{n-1} \cos\left(\frac{k\pi}{n}\right).$$

We show that U[regmod](n) is nonzero for every $n \geq 2$. More precisely, define the modified polynomials

$$\Phi[\text{regmod}]_n(x) = (x-1)^{-1}\, \Phi[\text{reg}]_n(x) = \prod_{\substack{d \parallel n \\ d>1}} \Phi_d(x).$$

Here, for example, $\Phi[\text{regmod}]_{12}(x) = \Phi_3(x)\Phi_4(x)\Phi_{12}(x) = x^8 + x^7 + x^6 + x^2 + x + 1$. All of the polynomials $\Phi[\text{regmod}]_n(x)$ have symmetric coefficients. By arguments similar to those leading to the formulas (23) and (24) we obtain the following identities.

Proposition 5.7.2 *For every $n \in \mathbb{N}$, $n \geq 2$,*

$$\mathrm{U[regmod]}(n) = \frac{\Phi[\mathrm{regmod}]_n(1)}{2^{\varrho(n)-1}} = \frac{\kappa(n)}{2^{\varrho(n)-1}},$$

and for every $n \in \mathbb{N}$, $n \geq 3$ odd,

$$\mathrm{V[regmod]}(n) = \frac{\Phi[\mathrm{regmod}]_n(-1)}{(-4)^{(\varrho(n)-1)/2}} = (-1/4)^{(\varrho(n)-1)/2}.$$

Note that $\varrho(n)$ is odd for every $n \in \mathbb{N}$ odd.

6. Maximal Orders of Certain Functions

Let $\sigma(n)$ be the sum of divisors of n and let $\psi(n) = n\prod_{p|n}(1+1/p)$ be the Dedekind function. The following open problems were formulated in [2]: What are the maximal orders of the functions $\varrho(n)\sigma(n)$ and $\varrho(n)\psi(n)$?

The answer is the following:

Proposition 6.1.

$$\limsup_{n\to\infty} \frac{\varrho(n)\sigma(n)}{n^2 \log\log n} = \limsup_{n\to\infty} \frac{\varrho(n)\psi(n)}{n^2 \log\log n} = \frac{6}{\pi^2}e^{\gamma},$$

where γ is the Euler-Mascheroni constant.

Proof. Apply the following general result, see [25, Cor. 1]: If f is a nonnegative real-valued multiplicative arithmetic function such that for each prime p,

i) $\rho(p) := \sup_{\nu\geq 0} f(p^{\nu}) \leq (1-1/p)^{-1}$, and

ii) there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ satisfying $f(p^{e_p}) \geq 1 + 1/p$,

then

$$\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^{\gamma} \prod_p \left(1 - \frac{1}{p}\right)\rho(p).$$

Take $f(n) = \varrho(n)\sigma(n)/n^2$. Here $f(p) = 1 + 1/p$ and $f(p^{\nu}) = 1 + 1/p^{\nu} + 1/p^{\nu+2} + 1/p^{\nu+3} + \ldots + 1/p^{2\nu} < 1 + 1/p$ for every prime p and every $\nu \geq 2$. This shows that $\rho(p) = 1 + 1/p$ and obtain that

$$\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^{\gamma} \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}e^{\gamma}.$$

The proof is similar for the function $g(n) = \varrho(n)\psi(n)/n^2$. In fact, $g(p) = f(p) = 1 + 1/p$ and $g(p^{\nu}) \leq f(p^{\nu})$ for every prime p and every $\nu \geq 2$, therefore the result for $g(n)$ follows from the previous one. □

Remark 6.2. Let $\sigma_s(n) = \sum_{d\,|\,n} d^s$. Then for every real $s > 1$,

$$\limsup_{n\to\infty} \frac{\varrho_s(n)\sigma_s(n)}{n^{2s}} = \frac{\zeta(s)}{\zeta(2s)}.$$

This follows by observing that for $f_s(n) = \varrho_s(n)\sigma_s(n)/n^{2s}$, $f_s(p) = 1 + 1/p^s$ and $f_s(p^{\nu}) = 1 + 1/p^{s\nu} + 1/p^{s(\nu+2)} + 1/p^{s(\nu+3)} + \ldots + 1/p^{2s\nu} < 1 + 1/p^s$ for every prime p and every $\nu \geq 2$. Hence, for every $n \in \mathbb{N}$,

$$f_s(n) \leq \prod_{p\,|\,n} \left(1 + \frac{1}{p^s}\right) < \prod_p \left(1 + \frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)},$$

and the lim sup is attained for $n = n_k = \prod_{1\leq j\leq k} p_j$ with $k \to \infty$, where p_j is the j-th prime.

References

[1] O. Alkam, E. A. Osba, On the regular elements in $\mathbb{Z}_n$, Turkish J. Math. 32 (2008) 31–39.
[2] B. Apostol, Extremal orders of some functions connected to regular integers modulo n, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. 21 (2013) 5–19.
[3] B. Apostol, L. Petrescu, On the number of regular integers modulo n, J. Algebra Number Theory Acad. 2 (2012) 337–351.
[4] M. Beck, M. Halloran, Finite trigonometric character sums via discrete Fourier analysis, Int. J. Number Theory 6 (2010) 51–67.
[5] H. Cohen, Number Theory, Vol. II. Analytic and Modern Tools, Graduate Texts in Mathematics 240, Springer, 2007.
[6] I. Ege, On defining the q-beta function for negative integers, Filomat 27 (2013) 251–260.
[7] I. Ege, E. Yýldýrým, Some generalized equalities for the q-gamma function, Filomat 26 (2012) 1227–1232.
[8] P. Haukkanen, L. Tóth, An analogue of Ramanujan's sum with respect to regular integers (mod r), Ramanujan J. 27 (2012) 71–88.
[9] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, (2rd edition), Graduate Texts in Mathematics 84, Springer, 1990.
[10] J.-M. de Koninck, I. Kátai, Some remarks on a paper of L. Toth, J. Integer Sequences 13 (2010) Article 10.1.2.
[11] G. Martin, A product of Gamma function values at fractions with the same denominator, Preprint 2009 arXiv:0907.4384v2.
[12] P. K. Menon, On the sum $\sum (a-1, n)[(a, n) = 1]$, J. Indian Math. Soc. (N.S.) 29 (1965) 155–163.
[13] P. J. McCarthy, Introduction to Arithmetical Functions, Springer, 1986.
[14] J. Morgado, Inteiros regulares módulo n, Gazeta de Matematica (Lisboa) 33 (1972) 1–5.
[15] J. Morgado, A property of the Euler φ-function concerning the integers which are regular modulo n, Portugal. Math. 33 (1974) 185–191.
[16] I. Niven, H. S. Zuckerman, H. L. Montgomery, An Introduction to the Theory of Numbers, (5rd edition), John Wiley & Sons, 1991.
[17] J. Sándor, L. Tóth, A remark on the gamma function, Elem. Math. 44 (1989) 73–76.
[18] J. Schulte, Über die Jordansche Verallgemeinerung der Eulerschen Funktion, Results Math. 36 (1999) 354–364.
[19] J. Singh, Defining power sums of n and $\varphi(n)$ integers, Int. J. Number Theory 5 (2009) 41–53.
[20] L. Tóth, Regular integers modulo n, Annales Univ. Sci. Budapest. Sect Comp. 29 (2008) 264–275.
[21] L. Tóth, A gcd-sum function over regular integers modulo n, J. Integer Sequences 12 (2009) Article 09.2.5.
[22] L. Tóth, A survey of gcd-sum functions, J. Integer Sequences 13 (2010) Article 10.8.1.
[23] L. Tóth, Menon's identity and arithmetical sums representing functions of several variables, Rend. Sem. Mat. Univ. Politec. Torino 69 (2011) 97–110.
[24] L. Tóth, Weighted gcd-sum functions, J. Integer Sequences 14 (2011) Article 11.7.7.
[25] L. Tóth, E. Wirsing, The maximal order of a class of multiplicative arithmetical functions, Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 353-364.
[26] D. Zhang, W. Zhai, Mean values of a gcd-sum function over regular integers modulo n, J. Integer Sequences 13 (2010) Article 10.4.7.
[27] D. Zhang, W. Zhai, Mean values of a class of arithmetical functions, J. Integer Sequences 14 (2011) Article 11.6.5.